11/26 Recitation Solutions

1. In class we showed that the SVD rank k approximation of a matrix A gives the best rank k approximation of A when measured using the Frobenius norm. It turns out that this is also true if we use the *spectral norm*. The spectral norm of a matrix M, denoted $||M||_S$, is the largest singular value of M. Consider the following SVD of a matrix A. Let

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 2 \\ -3 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{5} & 0 & 0 & 2/\sqrt{5} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -2/\sqrt{5} & 0 & 0 & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2\sqrt{5} & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \end{bmatrix}^T$$

   (a) Find the SVD rank 2 approximation A_2 of A.

$$A_2 = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T = 2\sqrt{3} \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 0 \\ -2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} + 4 \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 2\sqrt{3}/\sqrt{5} \\ -2 & -2 & 0 \\ 0 & 0 & -4\sqrt{3}/\sqrt{5} \end{bmatrix}$$

   (b) Calculate the Frobenius and spectral distances between A and A_2—i.e. $||A - A_2||_F$ and $||A - A_2||_S$ (Feel free to use Julia).

   The Frobenius distance is $||A - A_2||_F = \left|\left| \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 - 2\sqrt{3}/\sqrt{5} \\ -1 & 1 & 0 \\ 0 & 0 & -4 + 4\sqrt{3}/\sqrt{5} \end{bmatrix} \right|\right|_F \approx 2.24$

   Using Julia, we see that the first singular value of $A - A_2$ is 4. Hence the singular distance is $||A - A_2||_S = 2$.

   (c) Create some random matrices A in Julia (using the "randn(A)" command) and compute their Frobenius and spectral norms. Can you say anything in general about their relationship?

   The spectral norm is always less than or equal to the Frobenius norm.

2. Visit `https://timbaumann.info/svd-image-compression-demo/` and play around with the demo. It provides various images of different sizes and provides compressed images using SVD rank k approximations that you can compare using a slider. For each image, for what value of k is the image fairly clear? What does this say about the singular values σ_i, where $i > k$?

   For the first image, when $k = 100$, the image is fairly clear. This says that the singular values σ_i where $i > 100$ are close to zero.

3. Work though the **Julia14PCA** notebook for a PCA data visualization example.

   Solution in Julia notebook.